Measurement of Turbulent Density Fluctuations by Crossed Beam Correlation

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Crossed beam correlation (CBC) is an optical diagnostic for the fluctuation intensity of refractive index in fluid flows. Density fluctuation intensity may be inferred if the local chemical composition is known. Here, the theory and development of CBC are reviewed. Measurements in a stationary rocket plume are reported. Moderate agreement with predicted values of density fluctuation is obtained. Primary sources of experimental error are identified as statistical error in the estimation of correlation functions, and limited frequency and wave number response of the CBC instrument.

A critical analysis of CBC theory is presented in which the spatial resolution of the technique is characterized, and existing two- and three-beam CBC instruments are shown to be of limited accuracy in anisotropic turbulence. A revised six-beam CBC configuration is proposed.

Nomenclature

A	=	stationary random time dependent variable
a	=	nondimensional separation parameter, $\lvert\eta/L\rvert$
B	=	stationary random time dependent variable
b	=	integer
C	=	covariance (estimate)
c	=	integer
D	=	axial coordinate in plume
d	=	laser beam diameter
E_{ρ_1}	=	one-dimensional energy spectrum of ρ'
$E_{\rho\rho}$	=	three-dimensional energy spectrum of ρ'
E_ρ	=	isotropic energy spectrum of ρ'
e	=	error in covariance estimate [see Eq. (15)]
G	=	contribution to beam deflection covariance from point on one beam [see Eq. (33)]
g,h,i	=	generalized cartesian coordinates
H,I	=	generalized beam labels
J	=	Gladstone-Dale constant
j	=	output of autoregressive process
K	=	tangential coordinate in plume
k_1,k_2,k_3	=	wave number components
k_i	=	isotropic wave number $= (k_1{}^2 + k_2{}^2 + k_3{}^2)^{1/2}$
k_e	=	wave number characteristic of energy containing eddies
k_s	=	upper limit of inertial subrange
k_d	=	Kolmogoroff wave number
k_{ab}	=	detection limit due to finite beam thickness
k_{ae}	=	detection limit due to limited frequency response
L	=	integral scale of density fluctuations (subscript indicates orientation)
ℓ, m, N	=	integers
n	=	refractive index
$P(\xi, \eta, \zeta)$	=	RMS density fluctuation [see Eq. (30)]
p	=	integer
Q	=	scaling factor in model of E_ρ
q	=	anisotropic separation parameter [see Eqs. (9) and (35)]
R	=	correlation function [see Eq. (5)]
r	=	radial coordinate in plume
s	=	isotropic separation parameter $= (\xi^2 + \eta^2 + \zeta^2)^{1/2}$
T	=	time interval
t	=	time
U	=	random input to autoregressive process
u	=	mean velocity at beam intersection
W	=	weighting function $= G(\eta)/G(o)$
w	=	spatial resolution factor [see Eq. (39)]
X,Y,Z	=	beam labels
x,y,z	=	cartesian coordinates
α	=	autoregressive coefficient
β	=	characteristic nondimensional fluctuation of Gladstone-Dale constant
γ	=	covariance (exact)
Δt	=	sampling interval
ϵ	=	characteristic nondimensional density fluctuation
ζ, η	=	separation coordinates along beams
θ_{xy}	=	angular deflection in x-y plane of beam aligned with y axis
μ	=	ratio of characteristic scales of ρ' and mean flow
ξ	=	separation coordinate perpendicular to beam pair
ρ	=	density
σ	=	standard deviation
σ_e	=	standard error
σ_{ea}	=	asymptotic value for σ_e for $\Delta_t \rightarrow \infty$
τ	=	time delay
ϕ	=	angular coordinate

Superscripts

$\bar{}$	=	time average
$'$	=	fluctuating component
*	=	corrected to farfield static pressure

Subscripts

1	=	beam intersection point when $\xi = 0$
δ	=	plume exit plane

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0	= farfield conditions
m	= location of maximum $\overline{\rho'^2}$

Introduction

THEORETICAL predictions of turbulent chemically reacting flows are sensitive to assumptions concerning the level of fluctuations of scalar variables, such as density and temperature, so there is a need for experimental data to test modeling assumptions. The measurement of fluctuating scalars in the high-temperature, high-velocity exhaust plumes from gas turbines and rocket motors poses daunting technical problems. In most cases, the combination of excessive stagnation temperature and pressure, together with problems of spatial and temporal resolution, preclude the use of the traditional probe techniques; therefore, much effort has recently been expended on the development of nonintrusive optical diagnostics for the measurement of scalar properties of gas flows. Instantaneous point and two-dimensional measurements of temperature and species concentration are now possible using CARS or Laser Induced Fluorescence and their related techniques (see, for example, Eckbreth[1] and Kychakoff et al.[2]), but both of these diagnostics require expensive and somewhat delicate optical apparatus that must be protected from the heat, vibration, and chemical contamination associated with hostile flows.

Crossed Beam Correlation (CBC) is an alternative technique that permits localized measurement of mean square density fluctuations, requires only robust low-cost optical components, is comparatively insensitive to vibration and chemical contamination, and requires no flow seeding.

The purposes of this paper are fourfold: firstly, to give a brief account of the theory and development of CBC; secondly, to report some CBC measurements performed in a rocket exhaust plume; thirdly, to present an analysis of experimental errors; and, finally, to examine critically some aspects of the underlying theory of CBC.

Theory and Development of CBC

The relationship between refractive index and gas density, which is central to CBC, was first described by Gladstone and Dale,[3] who showed that fluids obey the law:

$$n = 1 + J\rho \tag{1}$$

The value of the Gladstone-Dale constant depends upon the chemical composition of the fluid but is independent of both temperature and pressure. It follows from Eq. (1) that a flowfield containing turbulent density fluctuations will exhibit associated fluctuations of refractive index.

A narrow beam of light traversing such a flow in, for example, the z direction will be deflected by refractive index gradients normal to the direction of propagation. The path integrated angular deflection may be resolved into components in the x and y directions, $\theta_{xz}(t)$, $\theta_{yz}(t)$.
Wilson and Damkevala[4] have shown that the fluctuating components of the deflections are related to the integral of the density gradient along the beam path, e.g.,

$$\theta'_{xz} = \int_{PATH} J(z)\left(\frac{\partial\rho}{\partial x}\right)'(z)\mathrm{d}z \tag{2}$$

The use of two crossed beams to obtain localized information appears to have been first proposed by Fisher and Krause,[5] although their technique relied on absorption of the beams by gas molecules in the flow rather than beam deflection. Wilson and Damkevala[4] subsequently adopted the use of crossed beams for beam deflection measurements. More recently, Winarto and Davis[6] have employed a crossed beam technique to measure density fluctuations in a jet, from which they have estimated the fluctuation intensities of temperature and pressure, and Davis[7] has used a similar technique to obtain integral scales in a diffusion flame.

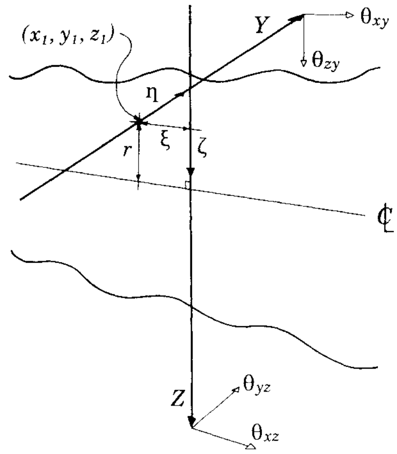

Fig. 1 Beam geometry for two-beam CBC in an axisymmetric jet.

The time mean product, or covariance, of the angular deflections of two intersecting beams may be regarded as a measure of those fluctuating density events that are able, simultaneously, to deflect both beams. If the density variations are induced by random turbulent mixing, the beams can only be deflected simultaneously by fluctuating density events that are convected through, or close to, the beam intersection point. Thus, the covariance can yield information about local flow properties at the intersection point.

Consider two beams, labeled Y and Z, parallel to the y and z axes respectively, passing through a region of turbulent variable density flow (see Fig. 1). From Eq. (2), the deflection covariance is given by

$$\overline{\theta'_{xy}\theta'_{xz}} = \int_{-\infty}^{\infty}\int \overline{J^2\left(\frac{\partial\rho}{\partial x}\right)'(x_1,y_1+\eta,z_1)\left(\frac{\partial\rho}{\partial x}\right)'(x_1+\xi,y_1,z_1+\zeta)}\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{3}$$

Variations in the Gladstone-Dale constant have been neglected.

Following Ball,[8] it may be shown that

$$\overline{\left(\frac{\partial\rho}{\partial x}\right)'(x_1,y_1+\eta,z_1)\left(\frac{\partial\rho}{\partial x}\right)'(x_1+\xi,y_1,z_1+\zeta)} = \left(\frac{\partial^2}{\partial x\,\partial\xi}-\frac{\partial^2}{\partial\xi^2}\right)\left(\overline{\rho'(x_1,y_1+\eta,z_1)\rho'(x_1+\xi,y_1,z_1+\zeta)}\right) \tag{4}$$

If the turbulent density field is assumed to be homogeneous, the density covariance will depend only upon the separation variables ξ,η,ζ, and, therefore, the operator $\partial^2/\partial x\,\partial\xi$ may be neglected.

A correlation function R may be defined as

$$R(x_1,y_1,z_1;\xi,\eta,\zeta) = \frac{\overline{\rho'(x_1,y_1+\eta,z_1)\rho'(x_1+\xi,y_1,z_1+\zeta)}}{[\overline{\rho'^2}(x_1,y_1+\eta,z_1)\overline{\rho'^2}(x_1+\xi,y_1,z_1+\zeta)]^{1/2}} \tag{5}$$

where R is unity when $\xi=\eta=\zeta=0$, and R tends to zero as any of the separation parameters, ξ,η,ζ, tend to infinity. If R is assumed to be approximately zero unless all the separation parameters are small, substitution into Eq. (3) from Eqs. (4) and (5) yields

$$\overline{\theta'_{xy}\theta'_{xz}} \simeq \bar{J}^2\,\overline{\rho'^2}(x_1,y_1,z_1)\int_{-\infty}^{\infty}\int \frac{-\partial^2}{\partial\xi^2}R(x_1,y_1,z_1;\xi,\eta,\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta \tag{6}$$

If the turbulent density field is isotropic, i.e., the correlation function is spherically symmetric with respect to the separation parameters, then one may write $R = R(x_1, y_1, z_1; s)$. Transposing Eq. (6) into polar coordinates, s, Φ, and setting $\xi = 0$ (beams intersecting) leads to

$$\overline{\theta'_{xy}\theta'_{xz}} \simeq \bar{J}^2 \, \overline{\rho'^2}(x_1, y_1, z_1) \int_0^{2\pi} \int_0^{\infty} \frac{-1}{s} \frac{\mathrm{d}R}{\mathrm{d}s}(x_1, y_1, z_1; s) s \, \mathrm{d}s \, \mathrm{d}\Phi \tag{7}$$

which yields on integration

$$\overline{\theta'_{xy}\theta'_{xz}} \simeq 2\pi \bar{J}^2 \, \overline{\rho'^2}(x_1, y_1, z_1) \tag{8}$$

Hence, the covariance of the beam deflections may be used to obtain the variance of the density field at the intersection point, provided that the turbulent density field is approximately isotropic.

Unfortunately, most flows of practical interest, especially those in which strong shear forces are acting, such as jets, will possess significantly anisotropic density fields. Kalghatgi[9] proposed that, in anisotropic turbulence, iso-correlation surfaces in ξ, η, ζ, space can be approximated as ellipsoids, so that one may write $R = R(x_1, y_1, z_1; q)$ where

$$q^2 = \frac{\xi^2}{L_x^2} + \frac{\eta^2}{L_y^2} + \frac{\zeta^2}{L_z^2} \tag{9}$$

Equation (8) then becomes for anisotropic turbulence

$$\overline{\theta'_{xy}\theta'_{xz}} \simeq 2\pi \bar{J}^2 \, \overline{\rho'^2}(x_1, y_1, z_1) \frac{L_y L_z}{L_x^2} \tag{10}$$

The new term $L_y L_z / L_x^2$ cannot be measured using a two-beam CBC technique. However, Kalghatgi[10] showed that the addition of a third beam, aligned with the x axis, would allow the determination of two extra beam deflection covariances:

$$\overline{\theta'_{yx}\theta'_{yz}} \simeq 2\pi \bar{J}^2 \, \overline{\rho'^2}(x_1, y_1, z_1) \frac{L_x L_z}{L_y^2} \tag{11a}$$

$$\overline{\theta'_{zx}\theta'_{zy}} \simeq 2\pi \bar{J}^2 \, \overline{\rho'^2}(x_1, y_1, z_1) \frac{L_x L_y}{L_z^2} \tag{11b}$$

and, therefore, by forming the geometric mean of the three measured covariances, the unknown scale ratio terms may be eliminated:

$$(\overline{\theta'_{xy}\theta'_{xz}} \; \overline{\theta'_{yx}\theta'_{yz}} \; \overline{\theta'_{zx}\theta'_{zy}})^{1/3} \simeq 2\pi \bar{J}^2 \, \overline{\rho'^2}(x_1, y_1, z_1) \tag{12}$$

Kalghatgi's[10] three-beam CBC instrument was designed for measurements in free jet flows. In order to avoid placing optical components within the flow, Kalghatgi[10] adopted the configuration shown in Fig. 2, in which the mean flow direction is perpendicular to the z axis, and at 45 deg to the x and y axes.

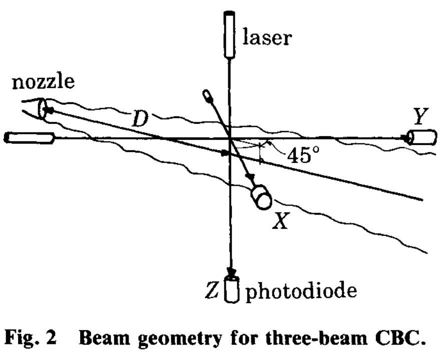

Fig. 2 Beam geometry for three-beam CBC.

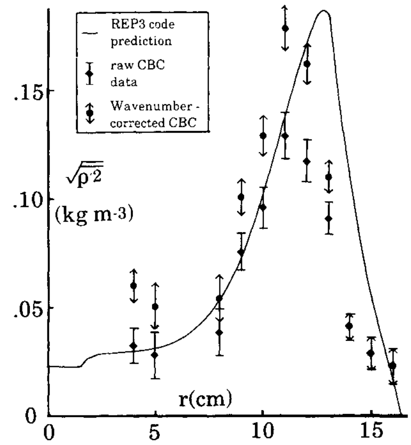

Fig. 3 Measured and predicted profiles of density fluctuation; $D = 0.33$ m.

Rocket Plume Measurements

The present authors have used Kalghatgi's three-beam configuration to probe the near-field exhaust plume of a stationary liquid-fueled rocket motor. The beams were provided by small Helium-Neon lasers (Coherent Radiation Model 80, giving 4 milliwatts at 632.8 nm). The $1/e^2$ beam diameter was approximately 0.8 mm. The beams were not focused. Typical beam path length within the plume was 0.25 m, and optical lever from intersection point to detector, 1.5 m. Beam deflections were measured using position sensitive photodiodes (United Detector Technology Model SC25, active area 18.5 mm square), which produced output signals proportional to the linear deflection of the laser spot centroid. All optical components were mounted in protective wooden enclosures and secured to a rigid steel frame surrounding the plume. The plume gases were exhausted from a 72.65 mm nozzle at approximately 2200 m/s and 2000 K. The plume was underexpanded, with an exit plane pressure of 4 atmospheres, which resulted in the formation of barrel shocks, and a consequent increase in aerodynamic noise. Noise levels close to the plume were estimated to have exceeded 140 dB. The oxidant employed was red fuming nitric acid, quantities of which were expelled from the nozzle as a fine mist at the end of each firing. These factors served to make the vicinity of the plume a very hostile environment for the use of measuring optics. Nevertheless, the apparatus survived approximately 200 rocket firings, typically of 10 s duration, without serious failures.

The deflection signals were recorded on analog tape, and, subsequently, a Hewlett Packard HP3721A Correlator was used to calculate the required deflection covariances. The Gladstone-Dale constant was calculated for each measurement location from predictions of time averaged chemical composition obtained from the REP3[11] computer code. Values of $\bar{J}$ varied from a maximum of 2.65×10^{-4} m^3 kg^{-1} on the plume centerline to 2.27×10^{-4} m^3 kg^{-1} in ambient air. The density fluctuation intensity was them obtained from Eq. (12).

Radial profiles of root mean square density fluctuation were obtained at a series of axial stations whose distance D from the nozzle exit plane covered the range 0.33 m $\leq D \leq$ 1.5 m (see Figs. 3-5). The solid lines of these figures are predicted profiles of $\sqrt{\overline{\rho'^2}}$, obtained from the REP3 computer program (see Kalghatgi et al.[11]).

Experimental points are plotted before and after adjustment for systematic error, due to limited wave number response (see following section). All points carry error bars indicating the estimated magnitude of random errors. Agreement between experimental results and theoretical predictions is significantly improved after systematic error adjustment, yielding disparities of less than 10% on peak levels. However, it should be noted that the model employed to estimate the magnitude of

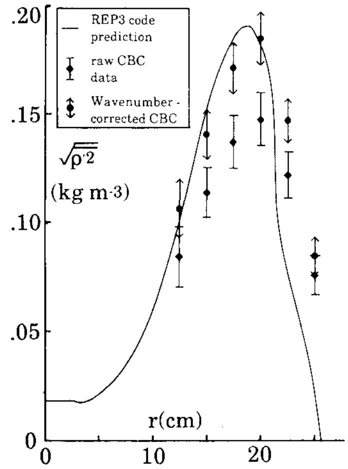

Fig. 4 Measured and predicted profiles of density fluctuation; $D = 1.0$ m.

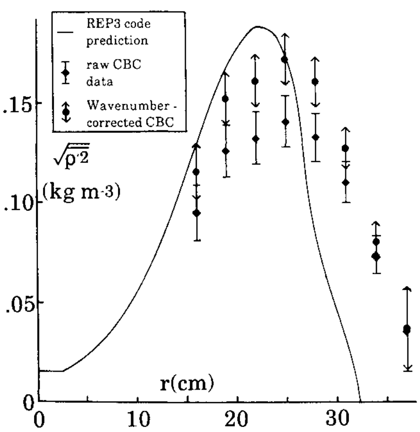

Fig. 5 Measured and predicted profiles of density fluctuation; $D = 1.5$ m.

the systematic error is very approximate, and, therefore, the adjusted data points can give only a semiquantitative indication of the effect of systematic errors on the experimental results.

The ratios of the scales L_x, L_y, L_z were obtained from the measured covariances by rearrangement of Eqs. (10) and (11). In general, it was found the $L_x \simeq L_y$, as would be expected in view of the symmetric disposition of the X and Y beams relative to the flow axis, whereas $0.6\ L_x < L_z < L_x$, indicating a significant degree of anisotropy within the turbulent density field.

Experimental Errors

Random Errors

The principal sources of random error in the measurement of $\sqrt{\overline{\rho'^2}}$ have been identified by Ball.[8] These included uncertainty in the optical levers ($\pm 0.3\%$), in photodiode amplifier calibrations ($\pm 5\%$), in Gladstone-Dale constant ($\pm 7.5\%$), and errors incurred in estimating the beam deflection covariances from data records of finite length. This latter source of error is analyzed at length below.

The beam deflection signals are random functions of time that may be assumed to be statistically stationary under steady-state flow conditions. For any pair of generalized stationary random time dependent variables $A(t)$, $B(t)$, the covariance function $\gamma_{AB}(\tau)$ is given by

$$\gamma_{AB}(\tau) = {}^{LIMIT}T \to \infty \; \frac{1}{T}\int_0^T A'(t)B'(t-\tau)\mathrm{d}t \tag{13}$$

where τ represents a time delay of signal B relative to A. Clearly it is impossible to measure $\gamma_{AB}(\tau)$ exactly, since no real experiment can be of infinite duration. If Eq. (13) is evaluated for some finite interval, then an estimate $C_{AB}(\tau) \simeq \gamma_{AB}(\tau)$ is obtained. The time histories of A' and B' over the finite interval represents only one of an infinite number of possible realizations of the random processes A,B.

If $C_{AB}(\tau)$ were to be evaluated p times, yielding $C_{AB}(\tau;0)\ldots C_{AB}(\tau;p-1)$, then it is to be expected that

$$\frac{1}{p}\sum_{m=0}^{p-1} C_{AB}(\tau;m) \simeq \gamma_{AB}(\tau) \tag{14}$$

The individual determinations $C_{AB}(\tau;m)$ would be distributed about the mean with some variance denoted $\mathrm{Var}[C_{AB}(\tau)]$. Taking a single value $C_{AB}(\tau)$ as a measurement of the covariance $\gamma_{AB}(\tau)$ will, therefore, produce an error $e_{AB}(\tau)$ where

$$e_{AB}(\tau) = \gamma_{AB}(\tau) - C_{AB}(\tau) \tag{15}$$

The error $e_{AB}(\tau)$ cannot be measured exactly, but its magnitude may be estimated from a knowledge of the variance of the covariance estimate, $\mathrm{Var}[C_{AB}(\tau)]$.

Here, covariance estimates have been obtained using a digital correlator. The variables A and B are sampled simultaneously N times at intervals of Δt, yielding $A(\ell)$, $B(\ell)$, $0 < \ell < N-1$. The covariance estimate is obtained using the discrete form of Eq. (13):

$$C_{AB}(b) = \frac{1}{N-b}\sum_{\ell=0}^{N-b-1}\left(A(\ell+b)-\bar{A}\right)\left(B(\ell)-\bar{B}\right) \tag{16}$$

Following Jenkins and Watts,[12] the covariance of any two covariance estimates formed at different time delays, $b\Delta t$, $c\Delta t$, is given in discrete from by Bartlett's formula:

$$Cov[C_{AB}(b),C_{AB}(c)] \simeq \frac{1}{N}\sum_{m=-\infty}^{\infty}\Big[\gamma_{AA}(m)\gamma_{BB}(m+c-b) + \gamma_{AB}(m+c)\gamma_{BA}(m-b)\Big] \tag{17}$$

If $b = c$, then Eq. (17) gives the variance of the covariance estimate at delay $b\Delta t$. For the special case where $b = c = 0$, one obtains:

$$\mathrm{Var}\Big[C_{AB}(0)\Big] \simeq \frac{1}{N}\sum_{m=-\infty}^{\infty}\Big[\gamma_{AA}(m)\gamma_{BB}(m) + \gamma_{AB}(m)\gamma_{BA}(m)\Big] \tag{18}$$

In order to make further progress, it is necessary to introduce a model of the digitized sequences $A(\ell)$, $B(\ell)$. A simple model for a signal of finite bandwidth is the first-order autoregressive process, which may be defined in discrete form[12] as

$$j(\ell) - \bar{j} = \alpha\left(j(\ell-1)-\bar{j}\right) + U(\ell) \tag{19}$$

The fluctuating component of the series $j(\ell)$ depends upon the previous value $j(\ell-1)$, weighted by the autoregressive coefficient α, $0 \le \alpha \le 1$, and upon a random input $U(\ell)$, where $\bar{U} = 0$. The autocovariance function of this process is given by

$$\gamma_{jj}(b) = \alpha^b\gamma_{jj}(0); \qquad b \ge 0 \tag{20}$$

Covariance estimates for the measurements reported here were obtained from signals replayed from tape at 1/16th real time and digitized at 1 kHz. Under these conditions, the digital autocovariance function estimates $C_{AA}(b)$ and $C_{BB}(b)$ are found to conform fairly closely to the form of Eq. (20) for b in the range $0 \le b \le \simeq 10$; therefore, the first-order autoregressive process has been used as an approximate model of laser deflection signals, for the purpose of error estimation.

The autoregressive coefficient a may be estimated by measuring the autocovariance function at $b = 0, 1$. Thus, from Eq. (20):

$$a = \frac{\gamma_{jj}(1)}{\gamma_{jj}(0)} \approx \frac{C_{jj}(1)}{C_{jj}(0)} \tag{21}$$

Using this model, Eq. (18) may be reduced[8] to

$$\mathrm{Var}\left[C_{j_1j_2}(0)\right] \simeq \left[\sigma_1^2\sigma_2^2 + C^2_{j_1j_2}(0)\right]\frac{1}{N}\left(\frac{1+a_1a_2}{1-a_1a_2}\right) \tag{22}$$

where

$$\sigma_1^2 = C_{j_1j_1}(0), \qquad \sigma_2^2 = C_{j_2j_2}(0)$$

The standard deviation of the distribution of covariance estimates, or standard error, is $\sigma_e = \{\mathrm{Var}[C_{j_1j_2}(0)]\}^{1/2}$. It has been assumed that this distribution is Gaussian, and, therefore, adopting a confidence interval of $\sim 95\%$, the error carried by an estimate of a deflection covariance is taken to be within the range $-2\sigma_e \leq e_{j_1j_2}(0) \leq 2\sigma_e$.

Ball[8] has shown that σ_e falls monotonically to an asymptote σ_{ea} as $\Delta t \to 0$. Selecting Δt, so that the product $a_1a_2 \simeq 0.3$ gives $\sigma_e/\sigma_{ea} \simeq 1.05$, and, therefore, little is gained by further reduction in Δt. The only other means of reducing σ_e is to increase the experimental duration. In many experimental situations, and in particular for rocket tests, the duration is limited by practical considerations, such as nozzle life, and, hence, these factors set limits to the attainable accuracy.

The estimated errors for each deflection covariance have been combined with estimates for the errors in the deflection signal calibrations, optical levers, and Gladstone-Dale constant, following the procedure described by Schenck,[13] in order to calculate error bars for each data point (shown on Figs. 3-5). Typical error margins are $\pm 15\%$.

Errors Due to Limited Wave Number Response

The crossed beam technique is limited in its response to high wave number components of the fluctuating density field by two factors: the finite thickness of the beams themselves and the limited frequency response of the photodetectors and associated electronics. As a consequence, if a significant proportion of the total turbulent energy of the density fluctuations occurs at wave numbers beyond the detection limit of the three-beam system, a systematic underestimation of $\sqrt{\overline{\rho'^2}}$ will result. The magnitude of this error may be estimated with the aid of a suitable model for the wave number spectrum of ρ'.

The density variance may be conveniently expressed, following Hinze,[14] in terms of the one-dimensional spectrum of ρ' in the k_1 (flow) direction:

$$\overline{\rho'^2} = \int_0^\infty E_{\rho 1}(k_1)\mathrm{d}k_1 \tag{23}$$

The one-dimensional spectrum $E_{\rho 1}(k_1)$ is defined as

$$E_{\rho 1}(k_1) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{\rho\rho}(k_1,k_2,k_3)\mathrm{d}k_2\mathrm{d}k_3 \tag{24}$$

where $E_{\rho\rho}(k_1,k_2,k_3)$ is the three-dimensional wave number spectrum of ρ'. Making the assumption that the turbulent density field is isotropic, one may introduce the "shell" spectrum $E_\rho(k_i)$, where

$$E_{\rho\rho}(k_1,k_2,k_3) = \frac{E_\rho(k_i)}{4\pi k_i^2}$$

and

$$k_i = (k_1^2 + k_2^2 + k_3^2)^{1/2}$$

Equation (24) then simplifies to

$$E_{\rho 1}(k_1) = \int_{k_1}^{\infty} \frac{E_\rho(k_i)}{k_i}\,\mathrm{d}k_i \tag{25}$$

and, therefore, combining Eqs. (23) and (25)

$$\overline{\rho'^2} = \int_0^\infty\int_{k_1}^\infty \frac{E_\rho(k_i)}{k_i}\mathrm{d}k_i\mathrm{d}k_1 \tag{26}$$

It will be assumed that a beam is deflected, without significant beam dispersion, by density gradient events whose minimum wavelength, measured in a plane perpendicular to the beam path, is $\sim 2\pi d$ or greater. In order for an event to contribute to the measured value of $\overline{\rho'^2}$, it must be detected by two orthogonal beams, and must, therefore, obey the condition that its wavelength is greater then $2\pi d$ in all directions. In terms of wave number, the criterion for detection is $0 \leq k_i \leq k_{ab}$, where $k_{ab} \simeq 1/d$. The estimated beam diameter at the intersection point is 2 mm; therefore, $k_{ab} \simeq 500$ rads m^{-1}.

The limited frequency response of the detectors and electronics results in a failure to detect density gradient events that are convected through the beams with an apparent frequency greater than 20 kHz. The apparent frequency is related to the k_1 component of the wave number vector by

$$f = k_1\frac{\overline{u}}{2\pi} \tag{27}$$

Therefore the criterion for detection, based upon electronic frequency response, is $0 \leq k_1 \leq k_{ae}$, where $k_{ae} = 2\pi \times 20{,}000/\overline{u}$.

The effect of limited wave number response upon the measured value of the density variance may be found by inserting k_{ab} and k_{ae} as the upper integration limits in Eq. (26).

In order to evaluate this effect, a very simple model for $E_\rho(k_i)$ was adopted:

$$\begin{aligned} E_\rho(k_i) &= Qk_i; \qquad 0 \leq k_i < k_e \\ &= Qk_e^{8/3}\,k_i^{-5/3}; \qquad k_e \leq k_i < k_s \\ &= Qk_e^{8/3}\,k_s^{16/3}\,k_i^{-7}; \qquad k_s \leq k_i < k_d \\ &= 0; \qquad k_i > k_d \end{aligned} \tag{28}$$

The constants $k_e^{8/3}$ and $k_s^{16/3}$ are introduced to make $E_\rho(k)$ continuous at k_e and k_s. The ratios k_e/k_d and k_s/k_d were estimated[8] as $\simeq 0.002$ and $\simeq 0.2$, respectively. The variation

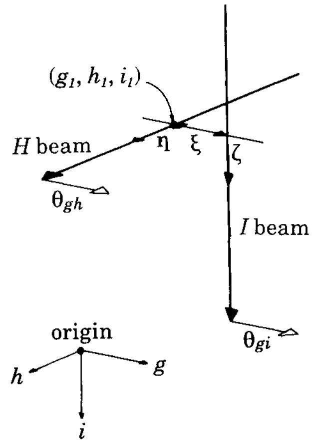

Fig. 6 Generalized two-beam coordinate system.

of k_d within the rocket plume was estimated with the aid of predictions of dissipation rate obtained from the REP3 plume modeling code.[11] Using the above model, the systematic error in the measured values of $\sqrt{\overline{\rho'^2}}$ was estimated for each data point. Figures 3-5 show experimental values before and after adjustment, using these error estimates.

It may be possible to reduce this source of error in the future by using waisted beams and electro-optic components with superior response at high frequencies. However, the efficiency of beam waisting is likely to be limited by the fluctuating refractive index field of the plume, the high wave number components of which tend to diverge the laser beams.

Critical Examination of CBC Theory

Generalized Two-Beam System

In order to analyze the theory of CBC, it is useful to define a generalized two-beam configuration that is independent of flowfield geometry, with beams labeled H and I, and cartesian axes g, h, i (see Fig. 6).

Noninformity of Chemical Composition Field

In obtaining Eq. (3), temporal variations in J have been neglected. This simplification is only strictly valid in chemically uniform flowfields, since J is a function of chemical composition. For the generalized two-beam system in chemically varying flow, Eq. (3) becomes

$$\overline{\theta'_{gh}\theta'_{gi}} = \int\int_{-\infty}^{\infty} \overline{\frac{\partial}{\partial g}\left[J(\eta)\rho(\eta)\right]'\frac{\partial}{\partial g}\left[J(\zeta)\rho(\zeta)\right]'}\,\mathrm{d}\eta\mathrm{d}\zeta \tag{29}$$

Ball[8] has expanded the integrand and performed an order of magnitude analysis, from which the most significant terms have been identified. These terms are listed below for the simplified case in which $\eta = \zeta = 0$:

	Term	Relative magnitude
1)	$\bar{J}^2\overline{\left(\frac{\partial\rho}{\partial g}\right)'^2}$	1
2)	$2\bar{J}\bar{\rho}\overline{\frac{\partial\rho'}{\partial g}\frac{\partial J'}{\partial g}}$	$\frac{2R\beta}{\epsilon}$
3)	$2\bar{J}\overline{J'\left(\frac{\partial\rho}{\partial g}\right)'^2}$	$2R\beta$
4)	$2\bar{J}\overline{\rho'\frac{\partial\rho'}{\partial g}\frac{\partial J'}{\partial g}}$	$2R\beta$

where R is a correlation coefficient

$$\epsilon = \frac{|\overline{\rho_\delta}^* - \overline{\rho_0}|}{\overline{\rho_\delta}^* + \overline{\rho_0}} \qquad \beta = \frac{|\overline{J_\delta} - \overline{J_0}|}{\overline{J_\delta} + \overline{J_0}}$$

For the rocket plume, these parameters have been estimated[8] at

$$R \simeq 0.1, \qquad \epsilon \simeq 0.8, \qquad \beta \approx 0.07$$

Hence, the sum of terms 1), 2), and 4) is expected to be approximately 4% of term 1); therefore, the effect of fluctuations in chemical composition would appear to be negligible. However, it should be noted that in hydrocarbon diffusion flames, values of β of the order of 0.5 may be encountered, due to the large differences in J between unburnt fuel and air; therefore, CBC measurements in this class of flow may be severely affected.

Inhomogeneity of the Turbulent Density Field

In deriving Eq. (6), the operator $\partial^2/\partial x\,\partial\zeta$ in Eq. (4) was discarded, on the assumption that the turbulent density field is homogeneous, i.e., that the spatial gradients of all time-averaged properties of the field are zero in all directions. For most flowfields of practical interest, this assumption is unrealistic; notably so for the rocket plume, where strong gradients of time mean properties exist. Retaining the equivalent operator, $\partial^2/\partial g\,\partial\xi$, for the generalized two-beam system Eq. (6) becomes

$$\begin{aligned}\overline{\theta'_{gh}\theta'_{gi}} = \int\int_{-\infty}^{\infty} \bar{J}^2\Bigg[& \overset{(1)}{-P(\eta)P(\xi,\zeta)\frac{\partial^2 R}{\partial\xi^2}} \\ & \overset{(2)}{-P(\eta)\frac{\partial P}{\partial g}(\xi,\zeta)\frac{\partial R}{\partial\xi}} + \overset{(3)}{\frac{\partial P}{\partial g}(\eta)\frac{\partial P}{\partial g}(\xi,\zeta)R} \\ & \overset{(4)}{+P(\eta)\frac{\partial P}{dg}(\xi,\zeta)\frac{\partial R}{\partial g}} \overset{(5)}{+P(\xi,\zeta)\frac{\partial P}{\partial g}(\eta)\frac{\partial R}{\partial\xi}} \\ & \overset{(6)}{+P(\eta)P(\xi,\zeta)\frac{\partial^2 R}{\partial g\,\partial\xi}}\Bigg]\,\mathrm{d}\eta\,\mathrm{d}\zeta\end{aligned} \tag{30}$$

where $P(\eta)$ and $P(\xi,\zeta)$ represent $\sqrt{\overline{\rho'^2}}\,(g_1, h_1+\eta, i_1)$ and $\sqrt{\overline{\rho'^2}}\,(g_1+\xi, h_1, i_1+\zeta)$, respectively. Terms (2) to (6) represent the effects of inhomogeneity in the density field. The importance of these terms relative to term (1) may be assessed by performing an order of magnitude analysis.

The magnitudes of differentials of R with respect to separation parameters are controlled by the form of the correlation function. Hence, the operator $\partial/\partial\xi$ is assigned a magnitude of $\approx 1/L$ when applied to R. The magnitude of the operator $\partial/\partial g$ is estimated as $\approx 1/r_m$. Writing $\mu = L/r_m$, the relative magnitudes of terms (1)-(6) are

Term	Relative magnitude
(1)	1
(2)	μ
(3)	μ^2
(4)	μ^2
(5)	μ
(6)	μ

A modified cross beam technique (Ball[8]) was used to approximately determine the transverse integral scale L_r in the rocket plume for $D = 330$ mm, $r/r_m = 1.09$, and $r_m = 110$ mm, yielding $L_r \simeq 4$ mm and, therefore, $\mu \simeq 0.036$. If this value of μ is assumed to be typical of the plume flowfield, then terms of order μ^2 may safely be neglected. The three terms of order μ each contain the factor $\partial R/\partial\xi$, which has been assigned a magnitude of $1/L$. However, in practice, the covariance is measured with the beams intersecting ($\xi = 0$), and if Kalghatgi's model is valid [Eq. (9)], then $\partial R/\partial\xi\,|_{\xi=0} = 0$, and these terms may also be neglected.

From the foregoing, it appears that inhomogeneities in the turbulent density field will only lead to significant errors in low Reynolds number flows, where the integral scale of the density fluctuations is of the same order as the characteristic dimensions of the flow, i.e., μ of order unity.

Spatial Resolution of CBC

The cross beam correlation technique is unusual among flow diagnostics in that, except for the case in which the integral scale of the density field is smaller than the beam diameters, it is the properties of the flowfield, and not those of the measuring system, that set limits upon the attainable spatial resolution.

In order to see why this is so, consider the covariance of deflections of the generalized two-beam system under the assumption of homogeneity:

$$\overline{\theta'_{gh}\theta'_{gi}} = -\bar{J}^2 \iint_{-\infty}^{\infty} P(\eta)P(\xi,\zeta)\frac{\partial^2 R}{\partial \xi^2}\,d\eta\, d\zeta \tag{31}$$

Wilson and Damkevala[4] have obtained an expression for the *RMS* density fluctuation at the intersection point $P(0)$ in terms of the deflection covariance by simplifying Eq. (31) to

$$\overline{\theta'_{gh}\theta'_{gi}} = -\bar{J}^2 P^2(0) \iint_{-\infty}^{\infty} \frac{\partial^2 R}{\partial \xi^2}\,d\eta\, d\zeta \tag{32}$$

The consequence of this approximation is that a value of P is attributed to the beam intersection point that is in reality an average of the values at all points on the H and I beams, weighted by the function

$$\frac{\partial^2 R}{\partial \xi^2}(g_1,h_1,i_1;\xi,\eta,\zeta)$$

Consider the contribution of some general point on the H beam to the deflection covariance, denoted by $G(\eta)$. From Eq. (31), $G(\eta)$ is given by

$$G(\eta) = \bar{J}^2 \int_{-\infty}^{\infty} P(\eta)P(\xi,\zeta)\frac{\partial^2 R}{\partial \xi^2}(\xi,\eta,\zeta)d\zeta \tag{33}$$

A weighting function $W(\eta)$ may be defined so that $W(\eta) = G(\eta)/G(0)$. This function compares the contribution of a general point to that of the point at the beam intersection, where $\eta = 0$. If gradients of P are modest, $W(\eta)$ may be approximated by

$$W(\eta) \simeq \frac{\int_{-\infty}^{\infty} \frac{\partial^2 R}{\partial \xi^2}(\xi,\eta,\zeta)\, d\zeta}{\int_{-\infty}^{\infty} \frac{\partial^2 R}{\partial \xi^2}(\xi,0,\zeta)\, d\zeta} \tag{34}$$

Adopting Kalghatgi's model, $R(\xi,\eta,\zeta)$ simplifies to $R(q)$, where, from Eq. (9)

$$q^2 = \frac{\xi^2}{L_g^2} + \frac{\eta^2}{L_h^2} + \frac{\zeta^2}{L_i^2} \tag{35}$$

The form of $R(q)$ is unknown. Hinze[14] indicates that in decaying isotropic turbulence, the correlation functions for velocity fluctuations are proportional to $\exp(-q^2)$. Assuming that the same proportionality is obeyed by $R(q)$, a suitable model for the correlation function is

$$R(q) = \exp\left(-\frac{\pi}{4}q^2\right) \tag{36}$$

The factor $\pi/4$ is included to normalize $R(q)$, so that

$$\int_0^{\infty} R(q)ds = 1 \tag{37}$$

Substituting for R in Eq. (34) from Eq. (36) and setting $\xi = 0$ (beams intersecting) yields:

$$W(\eta) = \exp\left(-\frac{\pi}{4}\frac{\eta^2}{L_h^2}\right) = R(0,\eta,0) \tag{38}$$

The fraction of the beam deflection covariance associated with points on the H beam within the range $-aL_h \leq \eta \leq aL_h$, denoted by $w(a)$, is given by

$$w(a) = \int_{-aL_h}^{aL_h} W(\eta)d\eta \Big/ \int_{-\infty}^{\infty} W(\eta)\, d\eta \tag{39}$$

On evaluating this expression, it is found that $w(k)$ reaches 0.9545 when $a = 2\sqrt{(2/\pi)}$, from which it may be concluded that approximately 95% of all contributions to the beam deflection covariance originates from points within a distance of $\pm 1.6L_h$ from the intersection point.

A similar analysis can be performed for beam I. The ranges for 95% covariance contribution may be regarded as a measure of the spatial resolution of the crossed beam technique. Since the integral scales L_h, L_i are properties of the turbulent density field, the spatial resolution of the technique is seen to be independent of the dimensions of the laser beams, provided that the integral scales are significantly larger than the beam radii.

For the case of the rocket plume, a typical transverse integral scale of the density field has been determined[8] as $\simeq 4$ mm, for $D = 330$ mm $r/r_m = 1.09$, yielding a "95% range" of ≈ 6.5 mm.

Modeling of the Density Correlation Function

The ability of the three beam CBC technique to determine the density variance in anisotropic turbulence relies heavily upon the validity of Kalghatgi's[9,10] model for the density correlation function [Eq. (9)], which holds that iso-correlation surfaces are ellipsoids with principal axes parallel to the x,y,z directions. A problem arises in that an ellipsoid possesses one unique set of principal axes; therefore, the model can, at best be valid for only three of the infinite set of possible orientations of the x,y,z axes relative to the flow. One may argue intuitively that, in an axisymmetric jet, these principal axes are most likely to be parallel to the axial radial and tangential directions (D,r,K). Since the x,y,z axes used for three beam CBC (Fig. 2) were not so aligned, some loss of accuracy must be expected in anisotropic turbulence.

To rectify this shortcoming, a revised beam geometry is proposed for future work. One would ideally like to align the beams with the principal axes, but such an arrangement is clearly impractical, since the laser head and detector for the axial beam would need to be at least partially immersed in the flow. However, it may be shown that the double integral appearing in the expression for the deflection covariance of a generalized two-beam system [Eq. (31)], although referring strictly only to pairs of points on the η and ζ axes, respectively, is approximately equivalent to an area integral in the η-ζ plane. The expression for the correlation coefficient R, in generalized coordinates, is

$$R = \frac{\overline{\rho'(g_1,h_1+\eta,\, i_1)\rho'(g_1+\xi,h_1,i_1+\zeta)}}{P(\eta)P(\xi,\zeta)} \tag{40}$$

but if $\partial R/\partial g$, $\partial R/\partial h$, $\partial R/\partial i$ are small, so that R is approximately independent of g,h,i over distances comparable with the integral scale L, as would be expected if $L \ll r_m$, then R may be written as

$$R \simeq \frac{\overline{\rho'(g_1,h_1,i_1)\rho'(g_1+\xi,h_1-\eta,i_1+\zeta)}}{P(0),P(\xi,-\eta,\zeta)} \tag{41}$$

in which case, if $\xi = 0$ (beams intersecting), the double integral of Eq. (31) becomes an area integral. Therefore, the value of the covariance is dependent upon the position of the beam intersection and the orientation of the g axis, but is independent of the orientations of the h,i axes. Hence, in order to measure the density variance in an anisotropic turbulent density field, it is only necessary to measure three covariances at

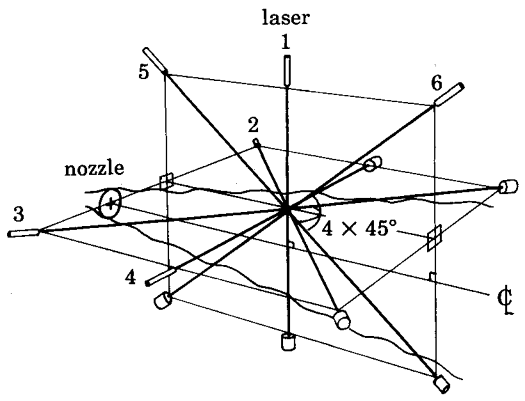

Fig. 7 Beam geometry for proposed six-beam CBC system.

a point in the flow such that the g axis is aligned with each of the three principal axes in turn. The required covariances could be obtained in an axisymmetric jet, using six intersecting beams (see Fig. 7). The measured covariances would be

Covariance	Direction of g axis
$\overline{\theta'_{D1}\theta'_{D4}}$	axial
$\overline{\theta'_{r2}\theta'_{r3}}$	radial
$\overline{\theta'_{K5}\theta'_{K6}}$	tangential

Provided that Kalghatgi's model is valid on the D,r,K axes, this configuration would permit a complete correction for the effects of anisotropic in the turbulent density field, while allowing all optical components to be positioned outside the flowfield.

Conclusions

The CBC technique has been shown to offer a relatively simple and inexpensive means of measuring turbulent density fluctuations in hostile flowfields. The successful application of CBC in a rocket plume has demonstrated the practical utility of the technique. The measured values of $\sqrt{\overline{\rho'^2}}$ are estimated to carry a random error of, typically, ±15%, arising principally from the determination of the beam deflection covariances from finite data records. Limited response to the high wave number components of the turbulent density field is predicted to have caused systematic underestimation of $\sqrt{\overline{\rho'^2}}$ in the rocket plume.

A critical examination of the theory of CBC has indicated that fluctuations in the chemical composition field and inhomogeneities in the turbulent density field are potential sources of error, but that for the rocket plume case such errors are negligible. The spatial resolution of CBC may be characterized by the "95% range," which is the distance from the intersection point along any beam, within which 95% of the contributions to the beam deflection covariance originate. The 95% range has been predicted to be ~1.6 × the integral scale of the density fluctuations.

Finally, it is argued that the orientation of the beams for three beam CBC, as proposed by Kalghatgi,[10] will not permit accurate measurements in anisotropic turbulence, and a revised configuration employing six beams is, therefore, proposed.

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